Choice over Time: Exponential Discounting

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Exponential discounting: When a person receives utility at different points in time, she seeks to maximize her *intertemporal utility*:

$$U \equiv u_1 + \delta u_2 + \delta^2 u_3 + \ldots + \delta^{T-1} u_T$$

or put another way:

$$= \sum_{t=1}^{T} \delta^{t-1} u_t.$$

- u_t is her **instantaneous utility** in period t (or her "well-being" in period t).
- δ is her **discount factor**, where $\delta \in (0, 1]$.

Instead of using

$$U = \sum_{t=1}^{T} \delta^{t-1} u_t$$

we sometimes use

$$U \equiv \sum_{t=1}^{T} \frac{1}{(1+\rho)^{t-1}} u_t.$$

- ρ is the person's **discount rate** (rate of time preference).
- Reminder: δ is the person's **discount factor**.
- Note: $\delta = 1/(1+\rho)$ or $\rho = (1/\delta) - 1$.

Example 1: Suppose you face the following choice:

(A) 90 utils in period 2 vs. (B) 160 utils in period 4

Which do you prefer?

Example 2: Suppose you have the opportunity to give up 5 utils now in order to gain 2 utils for each of the next three periods. Do you take it?

Example 3: Two-Period Saving-Consumption Decisions

Let c_1 be your consumption expenditures in period 1, and let c_2 be your consumption expenditures in period 2. Hence, in the end you must choose a consumption bundle (c_1, c_2).

You seek to maximize your intertemporal utility

$$U(c_1, c_2) = u(c_1) + \delta u(c_2).$$

- Note: $u(c_t)$ is your period- t instantaneous utility as a function of your period- t consumption --- typically assume $u' > 0$ and $u'' < 0$.

Let Y_1 be your income received in period 1, and let Y_2 be your income received in period 2.

Let r be the market interest rate, at which you can either save or borrow.

What is your budget constraint?

Suppose $Y_1 = W$ and $Y_2 = 0$ (*how-to-eat-a-cake* problem):

Budget constraint is

$$c_1 + \frac{c_2}{1+r} \leq W$$

Suppose instead $Y_1 > 0$ and $Y_2 > 0$.

Budget constraint is

$$c_1 + \frac{c_2}{1+r} \leq Y_1 + \frac{Y_2}{1+r}.$$

General Principle: If there are no liquidity constraints --- that is, if you can borrow and save at the same market interest rate --- then your budget constraint will take the form:

$$PDV \text{ of Consumption Expenditures } \leq PDV \text{ of Income Flows.}$$

So the problem becomes:

Choose (c_1, c_2) to maximize

$$U(c_1, c_2) = u(c_1) + \delta u(c_2)$$

subject to

$$c_1 + \frac{c_2}{1+r} \leq Y_1 + \frac{Y_2}{1+r} \equiv W.$$

We can do a graphical analysis.... (see board in class.)

Or we can solve explicitly for specific functional forms.

- For instance, when $U(c_1, c_2) = \ln c_1 + \delta \ln c_2$,

$$c_1 = \frac{W}{1+\delta} \qquad \text{and} \qquad c_2 = \frac{\delta(1+r)W}{1+\delta}.$$

(Solving this involves a simple application of constrained maximization; see board.)

What if there are liquidity constraints?

Consider Example 3, except suppose that when you borrow, you must pay an interest rate $r_B > r$.

- Budget constraint is:

$$c_1 + \frac{c_2}{1+r} \leq Y_1 + \frac{Y_2}{1+r} \equiv W_S \qquad \text{if } c_1 \leq Y_1$$

$$c_1 + \frac{c_2}{1+r_B} \leq Y_1 + \frac{Y_2}{1+r_B} \equiv W_B \qquad \text{if } c_1 > Y_1$$

Example 4: T -Period Saving-Consumption Decisions

Suppose that you consume in T different periods, and let c_t denote your consumption expenditures in period t. Hence, in the end you must choose a consumption bundle $(c_1, c_2, \ldots, c_T)$.

You seek to maximize your intertemporal utility

$$U(c_1, c_2, \ldots, c_T) = \sum_{t=1}^{T} \delta^{t-1} u(c_t).$$

- Let $(Y_1, Y_2, \ldots, Y_T)$ denote your income flows.
- Let r be the market interest rate, and assume no liquidity constraints.

Implication: You will choose $(c_1, c_2, \ldots, c_T)$ to maximize $U(c_1, c_2, \ldots, c_T)$ subject to

$$c_1 + \frac{c_2}{1+r} + \ldots + \frac{c_T}{(1+r)^{T-1}} \leq Y_1 + \frac{Y_2}{1+r} + \ldots + \frac{Y_T}{(1+r)^{T-1}}$$

Example 4 with $u(c) = \ln c$:

Goal: Choose $(c_1, c_2, \ldots, c_T)$ to maximize

$$U(c_1, c_2, \ldots, c_T) = \ln c_1 + \delta \ln c_2 + \ldots + \delta^{T-1} \ln c_T$$

subject to

$$c_1 + \frac{c_2}{1+r} + \ldots + \frac{c_T}{(1+r)^{T-1}} \leq W$$

where

$$W \equiv Y_1 + \frac{Y_2}{1+r} + \ldots + \frac{Y_T}{(1+r)^{T-1}}.$$

A relationships-between-variables approach

- It turns out that for each $t > 1$:

$$c_t = [\delta(1+r)]^{t-1} c_1$$

$$\text{or} \qquad c_t = \delta(1+r)c_{t-1}.$$

Solving for consumption:

$$c_1 = \frac{W}{1+\delta+\ldots+\delta^{T-1}} \quad \text{and} \quad c_t = \frac{[\delta(1+r)]^{t-1} W}{1+\delta+\ldots+\delta^{T-1}}.$$

Consider a simple two-period model of intertemporal choice. Suppose that Rocky receives income Y_1 in period 1 and additional income Y_2 in period 2. (He dies in his fight against Ivan Drago immediately after period 2). The market interest rate at which Rocky can both borrow and save is 4%. Finally, the Rocky's preferences are given by

$$U(c_1, c_2) = \frac{3}{2}(c_1)^{2/3} + \delta\frac{3}{2}(c_2)^{2/3}.$$

a) Derive the budget constraint that the person faces.

b) Solve for the optimal c_1 and c_2 as a function of δ .

c)(Bonus; do this later) When is the person saving in period 1? When is the person borrowing in period 1?

General Version of Discounted Utility

All of the ideas we have discussed about exponential utility actually fit under a broader umbrella.

We'll call this broader umbrella the *discounted-utility model*:

$$U^t = \sum_{x=0}^{T-t} D(x)\, u_{t+x}$$

- In this framework, U^t is intertemporal utility from perspective of period t.
- u_τ is instantaneous utility in period τ (or "well-being" in period τ).
- x is the delay before receiving some utility.
- $D(x)$ is a **discount function** that specifies the amount of discounting associated with delay x.

In principle, we could have any discount function. Exponential discounting **assumes** $D(x) = \delta^x$.

Key Features

(1) Impatience: For $\delta < 1$, $D(x)$ is monotonically declining in x.

- Put simply, longer delays imply more discounting.

(2) Constant discounting: For all x, $D(x+1)/D(x) = \delta$.

- This represents an even-handedness in how you view time.
- If we're thinking in terms of years, how you feel about this year vs. next year is the same as how you feel about next year vs. the following year is the same as how you feel about 5 years from now vs. 6 years from now.
- If we're thinking in terms of days, how you feel about today vs. tomorrow is the same as how you feel about tomorrow vs. the next day is the same as how you feel about 100 days from now vs. 101 days from now.

Key Features

(3) Time consistency: As time passes, you do not change your mind about the best course of action.

- Your relative preference between two calendar dates is independent of when you are asked (independent of your perspective).

Let's work through an example of how relative preferences depend (or do not depend) on one's perspective under exponential discounting....

From a period-1 perspective, where your intertemporal preferences are

$$U^1 = \sum_{x=0}^{T-1} D(x) u_{1+x}$$

how do you weight:

- period 2 vs. period 3?
- period 3 vs. period 5?

From a period-2 perspective, where your intertemporal preferences are%

$$U^2 = \sum_{x=0}^{T-2} D(x) u_{2+x}$$

how do you weight:

- period 2 vs. period 3?

Example 5: Suppose you have linear instantaneous utility, and that you must choose between the following two options:

- Option A: Receive payoff V_L in period 2.
- Option B: Receive payoff V_H in period 3.

In period 1, prefer Option B when . . . . . . ?

In period 2, prefer Option B when . . . . . . ?